October Math Gems

Problem of the week 5

§1 Problems

Problem 1.1. The expression $\frac{\sin\frac{\theta}{2}+\sin\theta}{1+\cos\frac{\theta}{2}+\cos\theta}$ equals

---

Problem 1.2. The minimum value of $\frac{1}{3\sin\theta-4\cos\theta+7}$ is

---

Problem 1.3. Find the area of the regular octagon inscribed in a circle of radius r

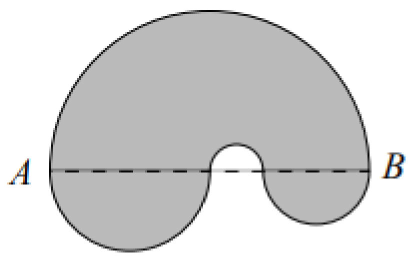
---

Problem 1.4. A quadratic polynomial $p(x)$ with real coefficients and leading coefficient 1 is called disrespectful if the equation $p(p(x)) = 0$ is satisfied by exactly three real numbers. Among all the disrespectful quadratic polynomials, there is a unique such polynomial $\tilde{p}(x)$ for which the sum of the roots is maximized. What is $\tilde{p}(1)$?

Problem 1.5. Prove that the following expression has no real solutions

$$\sqrt{2-x^2}+\sqrt[3]{3-x^3}=0$$

Problem 1.6. The boundary of the shaded figure consists of four semicircular arcs whose radii are all different. The centre of each arc lies on the line AB , which is 10 cm long. What is the length of the perimeter of the figure?



Problem 1.7. Solve the equation

$$\sin(x)-\sin(x)\cos(x)+\cos(x)=1$$

Problem 1.8. If $x, y \in \mathbb{R}$, what is the minimum value of

$$B(x,y)=x^2+2xy+2y^2+4y+2x-2017$$

and for what values of x and y is it achieved?

Problem 1.9. What is the value of

$$\frac{1}{10^{-9}+1}+\frac{1}{10^{-8}+1}+...+\frac{1}{10^{8}+1}+\frac{1}{10^{9}+1}$$

Problem 1.10. Simplify the given expression

$$10\sqrt{10\sqrt{10\sqrt{10\sqrt{10\sqrt{...}}}}}$$

Problem 1.11. If 6 people are seated around a circular table, what is the chance that two particular people always seated together ?

Problem 1.12. An ice-cream shop let you choose 2 out of 7 flavors at once and 1 out of 3 toppings. How many different icecreams can you make?

Problem 1.13. In $\triangle ABC$ let point D be the foot of the altitude from A to $\overline{BC}$. Suppose that $\angle A = 90°$, $AB - AC = 5$, and $BD - CD = 7$. Find the area of $\triangle ABC$

Problem 1.14. Let S be a sphere with radius 2. There are 8 congruent spheres whose centers are at the vertices of a cube, each has radius x, each is externally tangent to 3 of the other 7 spheres with radius x, and each is internally tangent to S. There is a sphere with radius y that is the smallest sphere internally tangent to S and externally tangent to 4 spheres with radius x. There is a sphere with radius z centered at the center of S that is externally tangent to all 8 of the spheres with radius x. Find $18x + 5y + 4z$.

Problem 1.15. Find the number of rearrangements of the nine letters AAABBBCCC where no three consecutive letters are the same. For example, count AABBCCABC and ACABBCCAB but not ABABCCCBA.

Problem 1.16. Starting at $12:00:00$ AM on January 1, 2022, after 13! seconds it will be y years (including leap years) and d days later, where $d < 365$. Find $y + d$.

Problem 1.17. A rectangle with width 30 inches has the property that all points in the rectangle are within 12 inches of at least one of the diagonals of the rectangle. Find the maximum possible length for the rectangle in inches.

Problem 1.18. Let a and b be positive integers satisfying $3a < b$ and $a^2 + ab + b^2 = (b+3)^2 + 27$. Find the minimum possible value of $a + b$.

Problem 1.19. Points X and Y lie on side $\overline{AB}$ of $\triangle ABC$ such that $AX = 20$, $AY = 28$, and $AB = 42$. Suppose $XC = 26$ and $YC = 30$. Find $AC + BC$.

Problem 1.20. There are real numbers x, y, and z such that the value of

$$x+y+z-\left(\frac{x^2}{5}+\frac{y^2}{6}+\frac{z^2}{7}\right)$$

reaches its maximum of $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n + x + y + z$.
